Davis-Putnam: A Combinatorial Approach

by Robert Cowen

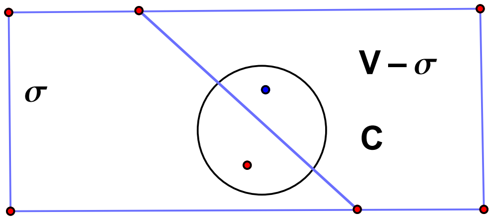
(Based on my papers: Generalized Davis-Putnam and satisfiability problems in mathematics , Logic Journal of the IGPL, to appear, and Davis-Putnam style rules for deciding Property S (with A. Kolany), Fund. Inform., 79(1-2), 5 - 15, published in 2007. These papers can be downloaded from my website: http://sites.google.com/site/robert-cowen/)

Property B

I. A set C of finite subsets of a set V will be said to have **Property B** if

1)there is a "hitting set" $\sigma \subset$V, for C that has a non-empty intersection with each element in C, $\sigma\bigcap$c$\neq\emptyset$, for each c$\in C$, and

2) no element of C is a subset of σ, c$\not\subseteq\sigma$ for each c$\in C$.



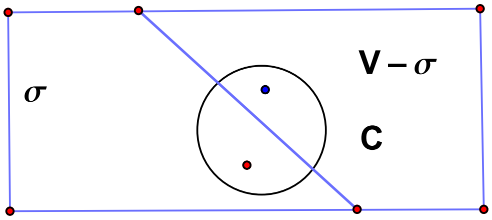
("B" is for F. Berstein who first considered this property in a paper in 1908.)

Property B, continued.

Equivalently,

II.There is a partition of V = $\sigma \bigcup (V - \sigma)$ such that $\sigma \bigcap c \neq \emptyset$ and $(V- \sigma)\bigcap c \neq \emptyset$, for each $c \in C$.

("V- σ" is the complement of σ in V.)



OR,

III.The vertices of hypergraph *C* can be 2-colored so that no hyper-edge is monochromatic.

(A **hypergraph** is a collection of finite subsets of a set; if the all the subsets are of size two, we have a "graph.")

Property S

Let V be a set and C and $\mathcal{F}$, sets of finite subsets of V, C and $\mathcal{F}$ not both empty. A set $\sigma\subset V$ will be said to **satisfy** the pair $<C,\mathcal{F}>$ if
1) σ is a hitting set for C, that is, $\sigma\bigcap c\neq\emptyset$, the empty set, for every $c\in C$,
2) $f\nsubseteq\sigma$ for each $f\in\mathcal{F}$.

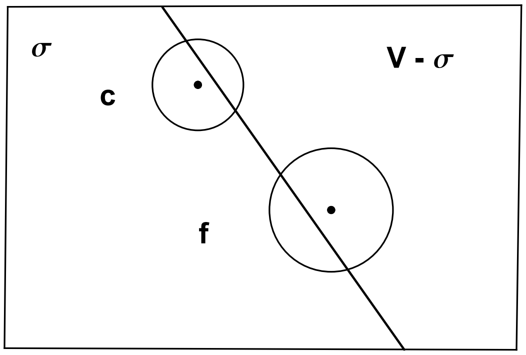
If there is some $\sigma\subset V$ which satisfies $<C,\mathcal{F}>$ we shall say that $<C,\mathcal{F}>$ has **Property S** or, briefly, that $<C,\mathcal{F}>$ is **satisfiable**.

We shall often refer to elements of C as "choice sets," and elements of $\mathcal{F}$ as "forbidden sets."
(If $C=\mathcal{F}$, we are back to Property B).

(" S" for A. Schrijver who studied this Property in the late 1970's in connection with set theoretic equivalents to the Prime Ideal Theorem.)

REMARKS
1. If C or $\mathcal{F}$ contains $\emptyset$, the empty set, then $<C,\mathcal{F}>$ is unsatisfiable.
2. If $C=\emptyset$ or $\mathcal{F}=\emptyset$, the empty set, then $<C,\mathcal{F}>$ is satisfiable.

Property S, continued.

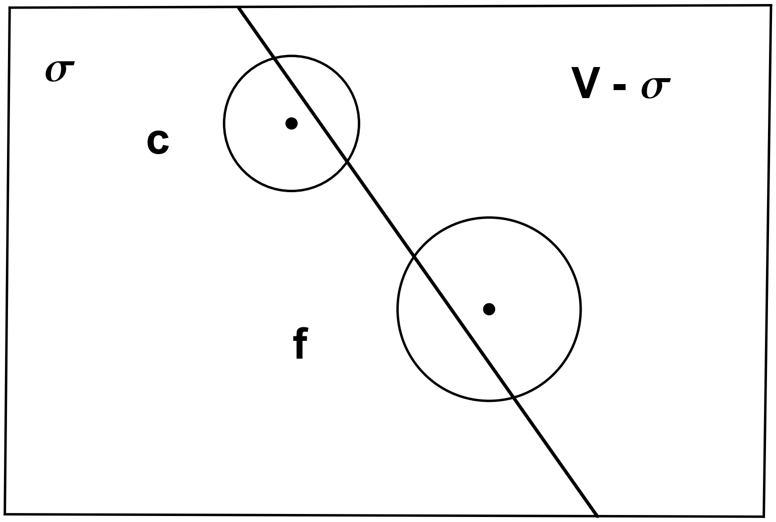


Equivalently,

<$\mathcal{C}$,$\mathcal{F}$> is **satisfiable** if there is a partition of V as $V = \sigma \bigcup (V - \sigma)$, where σ is a hitting set for $\mathcal{C}$ and $V - \sigma$, is a hitting set for $\mathcal{F}$.

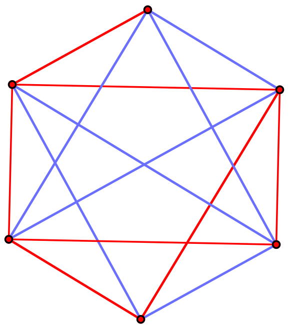
Duality

$< \mathcal{C}, \mathcal{F} >$ has Property S iff $< \mathcal{F}, \mathcal{C} >$ has Property S.



σ satisfies $< \mathcal{C}, \mathcal{F} >$ iff V-σ satisfies $<\mathcal{F}, \mathcal{C} >$.

Translating Mathematical Satisfiability Problems to Property S Problems



<u>Theorem</u>. Any red/blue coloring of the edges of the complete graph on 6 vertices will have a red or blue triangle.
We number the vertices, 1 to 6 and name edges with unordered pairs of their vertex numbers. Then

1) For each edge $\{i,j\}$ put $\{b_{\{i,j\}}, r_{\{i,j\}}\}$ into $\mathcal{C}$, $1 \leqslant i < j \leqslant 6$

2) For each triangle: $\{i,j\}$, $\{i,k\}$, $\{j,k\}$ put both $\{b_{\{i,j\}}, b_{\{i,k\}}, b_{\{j,k\}}\}$, $\{r_{\{i,j\}}, r_{\{i,k\}}, r_{\{j,k\}}\}$ into $\mathcal{F}$.

We must show that $<\mathcal{C}, \mathcal{F}>$ does **not** have Property S.

Logic (SAT Problem)

A **literal** is a statement letter or a negated statement letter.
A **clause** is a finite set of literals.
A set of clauses $\mathcal{C}$: {p, ¬q, r, ¬s}, {¬p, t, u, ¬v, ..., z}, ... is **satisfiable** if it is possible to pick a "literal" out of each without choosing a letter and its negation.

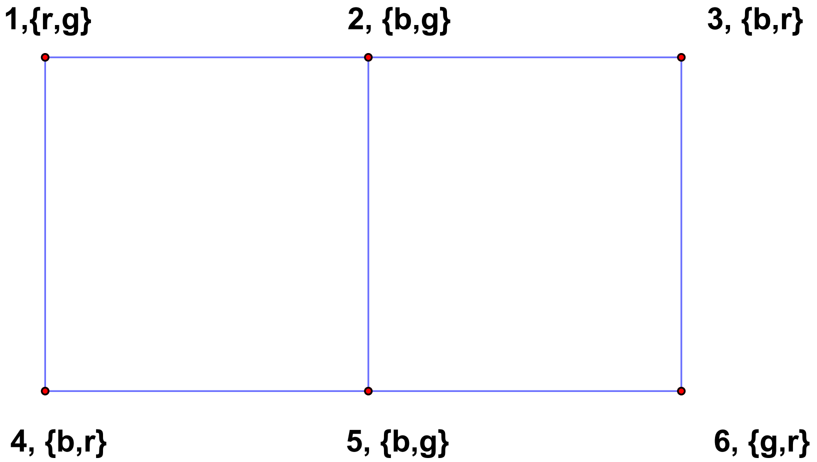
$\mathcal{C}$: {p, ¬q, r, ¬s}, {¬p, t, u, ¬v, ..., z}, ...

$\mathcal{F}$: {p, ¬p}, {q, ¬q}, {r, ¬r}, ...

$\mathcal{C}$ is satisfiable iff <$\mathcal{C}$, $\mathcal{F}$> has Property S.

◂ | ▸

List Vertex Coloring



A **list vertex coloring** is a proper vertex coloring of the graph with each color drawn from a prescribed list.

$C = \{\{r_1, g_1\}, \{b_2, g_2\}, \{b_3, r_3\}, \{b_4, r_4\}, \{b_5, g_5\}, \{g_6, r_6\}\}$.

$\mathcal{F} = \{\{r_1, r_4\}, \{g_1, g_2\}, \{g_2, g_5\}, \{b_2, b_3\}, \{b_2, b_5\}, \{b_4, b_5\}, \{g_5, g_6\}, \{r_3, r_6\}\}$.

Then the graph is list colorable iff $<C, \mathcal{F}>$ has Property S.

Sudoku

1			
	2		3
3			
	1		

1	3, 4	2, 4	2, 4
4	2	1	3
3	4	2	1
2	1	3, 4	4

a	b	c	d
e	f	g	h
i	j	k	l
m	n	o	p

The first diagram is a 4 by 4 sudoku puzzle; the second represents some additional work done; the third is a labeling for the boxes; so in the second diagram we see that the "b" box can have entries 3 or 4, etc. This suggests the following.

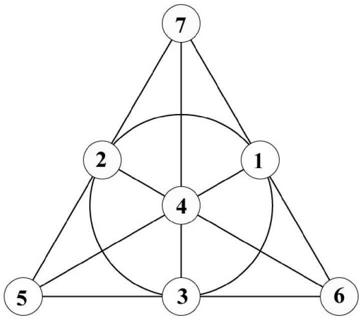
$C=\{\{a1\},\{b3,b4\},\{c2,c4\},\{d2,d4\},\{e4\},\{f2\},\{g1\},\{h3\},\{i3\},\{j4\},\{k2\},\{l1\},\{m2\},\{n1\},\{o3,o4\},\{p4\}\}$

$\mathcal{F}=\{\{b4,c4\},\{b4,d4\},\{c4,d4\},\{c2,d2\},\{o4,p4\},\{b4,j4\},\{c4,o4\},\{c2,k2\},\{d4,p4\},\{b4,e4\}\}$.

Then, the puzzle has a solution ⇔ $<C, \mathcal{F}>$ has Property S.

◄ | ►

Steiner Triple System, S_7



$C = S_7 = \{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{2,4,6\},\{2,5,7\},\{3,4,7\},\{3,5,6\}\}$.

Erdös (On a combinatorial problem, *Nordisk Mat. Tidskr.* 11 (1963), 5--10.) has shown that C does **not** have Property B. (The proof consisted of a calculation that was omitted)

Some Notation: "\", "\\"

If X = {{a, b, c, d}, {a, c}, {d}},

X\a = {{b, c, d}, {c}, {d }} (read "X drop a") and

X\\a = {{d}} (read "X double-drop a ").

Thus "\" removes a particular element from each set in a set of sets, while " \\ " removes every set containing that element from a set of sets.

◀ | ▶

The Davis/Putnam Rules

We shall mimic the widely used propositional logic rules for testing satisfiability as ennunciated in the early 1960s in the famous papers of Davis, Putnam and Davis, Longeman, Loveland(DPLL). There are three main rules.

1) **Unit Prune or Unit Propagation**: Drop any unit or one element clause, {p}({¬p}); also remove all clauses which contain that element; finally remove ¬p (p) from the remaining clauses. Then the original set is satisfiable iff the resulting set is.

2) **Pure Literal Rule**: If a set of clauses contains p but not ¬p, then drop all clauses containing p. Similarly, if set of clauses contains ¬p but not p, then drop all clauses containing ¬p. Then the original set is satisfiable iff the resulting set is.

3) **Splitting Rule**: Starting with clauses, $\mathcal{C}$, if p and ¬p are both present,
  a) Drop all clauses containing p, getting clausal set $\mathcal{C} \backslash\backslash p$.
  b) Drop all clauses containing ¬p, getting clausal set $\mathcal{C} \backslash\backslash \neg p$.
Then, $\mathcal{C}$ is satisfiable ⇔ ($\mathcal{C} \backslash\backslash p$ is satisfiable) or ($\mathcal{C} \backslash\backslash \neg p$ is satisfiable).

Combinatorial Davis-Putnam Rules

Unit Prune Rule. If some clause in C consists of a unit, $\{v\}$, then drop any clause in C containing the element v, resulting in clause set, $C\backslash\backslash v$; further, remove v from any forbidden set in $\mathcal{F}$ obtaining a new collection of forbidden sets, $\mathcal{F}\backslash v$. Then

Prop. The resulting pair, $<C\backslash\backslash v, \mathcal{F}\backslash v>$, is satisfiable $\Leftrightarrow$ $< C, \mathcal{F} >$ is satisfiable.

Idea. Surely since $\{v\}$ is a clause, v must belong to **any** satisfying hitting subset, σ. Then any clause containing v will also be "hit" by σ. Thus the question is "reduced" to showing that the clauses in $C\backslash\backslash v$ are also "hit."

Also, since $v \in \sigma$, the forbidden sets, f, can be replaced by those obtained by removing v from sets, $f \in \mathcal{F}$.

Often further unit clauses will be generated by this last operation in which case, by Duality, Unit Prune can be used again. In fact, Unit Prune alone can sometimes be used to solve the problem!

◀ | ▶

Solving the Sukoku Problem by Unit Prune Alone (Along with Duality)

C={{a1},{b3,b4},{c2,c4},{d2,d4},{e4},{f2},{g1},{h3},{i3},{j4},{k2},{l1},{m2},{n1},{o3,o4},{p4}}

$\mathcal{F}$= {{b4,c4},{b4,d4},{c4,d4},{c2,d2},{o4,p4},{b4,j4},{c4,o4},{c2,k2},{d4,p4},{b4,e4}}.

Then, by the Unit Prune Rule, we can drop any set containing an element belonging to a singleton and also remove their elements from $\mathcal{F}$. This leads to the following sets, C_1, $\mathcal{F}_1$.

C_1={{b3,b4},{c2,c4},{d2,d4},{o3,o4}}

$\mathcal{F}_1$= {{b4,c4},{b4,d4},{c4,d4},{c2,d2},{o4},{b4},{c2},{d4},{b4}}

Next, by Duality, we can use Unit Prune on the new unit clauses in $\mathcal{F}_1$, dropping all clauses which contain elements in unit clauses, to obtain, $\mathcal{F}_2$= {}, the empty set. If we remove these unit elements from C_1, we get C_2={{b3},{c4},{d2},{o3}}. Hence <C_2, $\mathcal{F}_2$> is satisfiable and thus so are <C_1, $\mathcal{F}_1$> and <C, $\mathcal{F}$>.

It is now not hard to see that the unit clauses in C, C_2 define the solution: a=1,b=3, c=4, d=2, e=4, f=2, g=1, h=3, i=3, j=4, k=2, l=1, m=2, n=1, o=3, p=4.

The Pure Literal Rule

Davis-Putnam Pure Literal Rule: If a propositional clause contains a literal while its opposite is missing from all the clauses, remove all clauses containing that literal. The result is satisfiable iff the orginal set of clauses was satisfiable.

Combinatorial Pure Literal Rule. Suppose $v \in \bigcup C$. Then v is a **pure literal** for $<C, \mathcal{F}>$ if for any $f \in \mathcal{F}$ with $v \in f$, there is some $w \in f$ with $w \notin \bigcup C$. Then, if v is a pure literal, $<C, \mathcal{F}>$ is satisfiable iff $<C \setminus\setminus v, \mathcal{F}>$ is satisfiable.

The idea here is that clauses containing pure literals can be removed from the clauses in C without effecting the satisfiability since we could have chosen these pure elements without fear of forming any forbidden sets when choosing elements from $\bigcup C$.

◂ | ▸

The Splitting Rule

Splitting Rule. Finally we turn to the "Splitting Rule." Suppose $v \in \bigcup C$ and σ satisfies $<\mathcal{C}, \mathcal{F}>$; if $v \in \sigma$, then surely σ satisfies $<\mathcal{C} \bigcup \{\{v\}\}, \mathcal{F}>$; if $v \notin \sigma$, $\{v\} \not\subseteq \sigma$, and this implies σ satisfies $<\mathcal{C}, \mathcal{F} \bigcup \{\{v\}\}>$. Also, it should be clear, that if either σ satisfies $<\mathcal{C} \bigcup \{\{v\}\}, \mathcal{F}>$ or σ satisfies $<\mathcal{C}, \mathcal{F} \bigcup \{\{v\}\}>$, then σ satisfies $<\mathcal{C}, \mathcal{F}>$. Hence we have the following principle.

Generalized Splitting Rule, I. Suppose that $v \in \bigcup C$. Then $<\mathcal{C}, \mathcal{F}>$ is satisfiable if and only if either $<\mathcal{C} \bigcup \{\{v\}\}, \mathcal{F}>$ or $<\mathcal{C}, \mathcal{F} \bigcup \{\{v\}\}>$ is satisfiable.

Combining this with Unit Prune and Duality gives, gives the following version.

Generalized Splitting Rule II. Suppose $v \in \bigcup C$. Then $<\mathcal{C}, \mathcal{F}>$ is satifiable if and only if either $<C \backslash\backslash v, \mathcal{F} \backslash v>$ or $<\mathcal{C} \backslash v, \mathcal{F} \backslash\backslash v>$ is satisfiable.

We now show the List Coloring example isn't satisfiable. Recall,

$C= \{\{r_1 ,g_1\},\{b_2 ,g_2\},\{b_3,r_3\},\{b_4,r_4\},\{b_5,g_5\},\{g_6,r_6\}\}.$
$\mathcal{F}=\{\{r_1, r_4\},\{g_1, g_2\},\{g_2, g_5\},\{b_2 ,b_3\}, \{b_2,b_5\}, \{b_4,b_5\},\{g_5,g_6\}, \{r_3,r_6\}\}.$

We first use the splitting rule, with b_2 as the splitting variable. We need to show that both $<C\backslash\backslash b_2,\mathcal{F}\backslash b_2>$ and $<\mathcal{F}\backslash\backslash b_2,C\backslash b_2>$ are both unsatisfiable; we only consider the first case here. Let $C_1 = C \backslash\backslash b_2$; $\mathcal{F}_1 = \mathcal{F} \backslash b_2$.

$C_1 = \{\{r_1, g_1\}, \{b_3, r_3\}, \{b_4, r_4\}, \{b_5, g_5\}, \{g_6, r_6\}\}$

$\mathcal{F}_1 = \{\{r_1, r_4\}, \{g_1, g_2\}, \{g_2, g_5\},$
$\{b_3\}, \{b_5\}, \{b_4, b_5\}, \{g_5, g_6\}, \{r_3, r_6\}\}.$

We then apply Unit Prune with the units $\{b_3\}, \{b_5\}$ in $\mathcal{F}_1$, getting,
$C_2 = \{\{r_1 , g_1\}, \{r_3\}, \{b_4, r_4\}, \{g_5\}, \{g_6, r_6\}\}$
$\mathcal{F}_2 = \{\{r_1, r_4\}, \{g_1, g_2\}, \{g_2, g_5\}, \{g_5, g_6\}, \{r_3, r_6\}\}$

If we apply Unit Prune with the units $\{g_5\}, \{r_3\}$ in C_2 we get
$C_3 = \{\{r_1 , g_1\}, \{b_4, r_4\}, \{g_6, r_6\}\}$
$\mathcal{F}_3 = \{\{r_1, r_4\}, \{g_1, g_2\}, \{g_2\}, \{b_4, b_5\}, \{g_6\}, \{r_6\}\}.$
Finally the units $\{g_6\}, \{r_6\}$ in $\mathcal{F}_3$ yield {} when applied to C_3. Unsatisfiable!

◄ | ►

Ramsey Theory Example

Here we must show that <*C*,*F*> does not have Property S, where,

$$
\begin{aligned}
\mathcal{C} = \{ & \mathbf{b}_{\{1,2\}}, \mathbf{r}_{\{1,2\}}\}, \{\mathbf{b}_{\{1,3\}}, \mathbf{r}_{\{1,3\}}\}, \{\mathbf{b}_{\{1,4\}}, \mathbf{r}_{\{1,4\}}\}, \\
& \{\mathbf{b}_{\{1,5\}}, \mathbf{r}_{\{1,5\}}\}, \{\mathbf{b}_{\{1,6\}}, \mathbf{r}_{\{1,6\}}\}, \{\mathbf{b}_{\{2,3\}}, \mathbf{r}_{\{2,3\}}\}, \\
& \{\mathbf{b}_{\{2,4\}}, \mathbf{r}_{\{2,4\}}\}, \{\mathbf{b}_{\{2,5\}}, \mathbf{r}_{\{2,5\}}\}, \{\mathbf{b}_{\{2,6\}}, \mathbf{r}_{\{2,6\}}\}, \\
& \{\mathbf{b}_{\{3,4\}}, \mathbf{r}_{\{3,4\}}\}, \{\mathbf{b}_{\{3,5\}}, \mathbf{r}_{\{3,5\}}\}, \{\mathbf{b}_{\{3,6\}}, \mathbf{r}_{\{3,6\}}\}, \\
& \{\mathbf{b}_{\{4,5\}}, \mathbf{r}_{\{4,5\}}\}, \{\mathbf{b}_{\{4,6\}}, \mathbf{r}_{\{4,6\}}\}, \{\mathbf{b}_{\{5,6\}}, \mathbf{r}_{\{5,6\}}\}\}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{F} = \{ & \{\mathbf{b}_{\{1,2\}}, \mathbf{b}_{\{1,3\}}, \mathbf{b}_{\{2,3\}}\}, \{\mathbf{b}_{\{1,2\}}, \mathbf{b}_{\{1,4\}}, \mathbf{b}_{\{2,4\}}\}, \\
& \{\mathbf{b}_{\{1,2\}}, \mathbf{b}_{\{1,5\}}, \mathbf{b}_{\{2,5\}}\}, \{\mathbf{b}_{\{1,2\}}, \mathbf{b}_{\{1,6\}}, \mathbf{b}_{\{2,6\}}\}, \\
& \{\mathbf{b}_{\{1,3\}}, \mathbf{b}_{\{1,4\}}, \mathbf{b}_{\{3,4\}}\}, \{\mathbf{b}_{\{1,3\}}, \mathbf{b}_{\{1,5\}}, \mathbf{b}_{\{3,5\}}\}, \\
& \{\mathbf{b}_{\{1,3\}}, \mathbf{b}_{\{1,6\}}, \mathbf{b}_{\{3,6\}}\}, \{\mathbf{b}_{\{1,4\}}, \mathbf{b}_{\{1,5\}}, \mathbf{b}_{\{4,5\}}\}, \\
& \{\mathbf{b}_{\{1,4\}}, \mathbf{b}_{\{1,6\}}, \mathbf{b}_{\{4,6\}}\}, \{\mathbf{b}_{\{1,5\}}, \mathbf{b}_{\{1,6\}}, \mathbf{b}_{\{5,6\}}\}, \\
& \{\mathbf{b}_{\{2,3\}}, \mathbf{b}_{\{2,4\}}, \mathbf{b}_{\{3,4\}}\}, \{\mathbf{b}_{\{2,3\}}, \mathbf{b}_{\{2,5\}}, \mathbf{b}_{\{3,5\}}\}, \\
& \{\mathbf{b}_{\{2,3\}}, \mathbf{b}_{\{2,6\}}, \mathbf{b}_{\{3,6\}}\}, \{\mathbf{b}_{\{2,4\}}, \mathbf{b}_{\{2,5\}}, \mathbf{b}_{\{4,5\}}\}, \\
& \{\mathbf{b}_{\{2,4\}}, \mathbf{b}_{\{2,6\}}, \mathbf{b}_{\{4,6\}}\}, \{\mathbf{b}_{\{2,5\}}, \mathbf{b}_{\{2,6\}}, \mathbf{b}_{\{5,6\}}\}, \\
& \{\mathbf{b}_{\{3,4\}}, \mathbf{b}_{\{3,5\}}, \mathbf{b}_{\{4,5\}}\}, \{\mathbf{b}_{\{3,4\}}, \mathbf{b}_{\{3,6\}}, \mathbf{b}_{\{4,6\}}\}, \\
& \{\mathbf{b}_{\{3,5\}}, \mathbf{b}_{\{3,6\}}, \mathbf{b}_{\{5,6\}}\}, \{\mathbf{b}_{\{4,5\}}, \mathbf{b}_{\{4,6\}}, \mathbf{b}_{\{5,6\}}\}, \\
& \{\mathbf{r}_{\{1,2\}}, \mathbf{r}_{\{1,3\}}, \mathbf{r}_{\{2,3\}}\}, \{\mathbf{r}_{\{1,2\}}, \mathbf{r}_{\{1,4\}}, \mathbf{r}_{\{2,4\}}\}, \\
& \{\mathbf{r}_{\{1,2\}}, \mathbf{r}_{\{1,5\}}, \mathbf{r}_{\{2,5\}}\}, \{\mathbf{r}_{\{1,2\}}, \mathbf{r}_{\{1,6\}}, \mathbf{r}_{\{2,6\}}\}, \\
& \{\mathbf{r}_{\{1,3\}}, \mathbf{r}_{\{1,4\}}, \mathbf{r}_{\{3,4\}}\}, \{\mathbf{r}_{\{1,3\}}, \mathbf{r}_{\{1,5\}}, \mathbf{r}_{\{3,5\}}\}, \\
& \{\mathbf{r}_{\{1,3\}}, \mathbf{r}_{\{1,6\}}, \mathbf{r}_{\{3,6\}}\}, \{\mathbf{r}_{\{1,4\}}, \mathbf{r}_{\{1,5\}}, \mathbf{r}_{\{4,5\}}\}, \\
& \{\mathbf{r}_{\{1,4\}}, \mathbf{r}_{\{1,6\}}, \mathbf{r}_{\{4,6\}}\}, \{\mathbf{r}_{\{1,5\}}, \mathbf{r}_{\{1,6\}}, \mathbf{r}_{\{5,6\}}\}, \\
& \{\mathbf{r}_{\{2,3\}}, \mathbf{r}_{\{2,4\}}, \mathbf{r}_{\{3,4\}}\}, \{\mathbf{r}_{\{2,3\}}, \mathbf{r}_{\{2,5\}}, \mathbf{r}_{\{3,5\}}\}, \\
& \{\mathbf{r}_{\{2,3\}}, \mathbf{r}_{\{2,6\}}, \mathbf{r}_{\{3,6\}}\}, \{\mathbf{r}_{\{2,4\}}, \mathbf{r}_{\{2,5\}}, \mathbf{r}_{\{4,5\}}\}, \\
& \{\mathbf{r}_{\{2,4\}}, \mathbf{r}_{\{2,6\}}, \mathbf{r}_{\{4,6\}}\}, \{\mathbf{r}_{\{2,5\}}, \mathbf{r}_{\{2,6\}}, \mathbf{r}_{\{5,6\}}\}, \\
& \{\mathbf{r}_{\{3,4\}}, \mathbf{r}_{\{3,5\}}, \mathbf{r}_{\{4,5\}}\}, \{\mathbf{r}_{\{3,4\}}, \mathbf{r}_{\{3,6\}}, \mathbf{r}_{\{4,6\}}\}, \\
& \{\mathbf{r}_{\{3,5\}}, \mathbf{r}_{\{3,6\}}, \mathbf{r}_{\{5,6\}}\}, \{\mathbf{r}_{\{4,5\}}, \mathbf{r}_{\{4,6\}}, \mathbf{r}_{\{5,6\}}\}\}
\end{aligned}
$$

We ran our program on a laptop computer and it took about 0.61 seconds. If we use the standard combinatorial argument to argue that three of the five edges connected to vertex 1 must be the same color and thus replace the first three clauses of C by singletons, the program runs in 0.04 seconds (and could be done "by hand". Try it!)

Final Remarks

1. The rules given are easy to program and we are suggesting that computer programs for Satisfiability Problems be written in this general Property S setting using our Davis/Putnam rules rather than translate into Propositional Logic and using the Davis/Putnam rules there. This will bring the programming much closer to the problem being considered and perhaps suggest problem-specific rules which can speed computations.

2. The Duality Principle seems a useful addition in testing for Property S. Can other general and efficient rules be found for satisfiablility testing of Property S?