Smullyan's Analytic Tableaux Applied to Probabilistic Entailment in Jøsang's Subjective Logic (and Related Systems)

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This Conference Needs No Title: A 90th Birthday Celebration Honoring Raymond Smullyan

CUNY Graduate Center,
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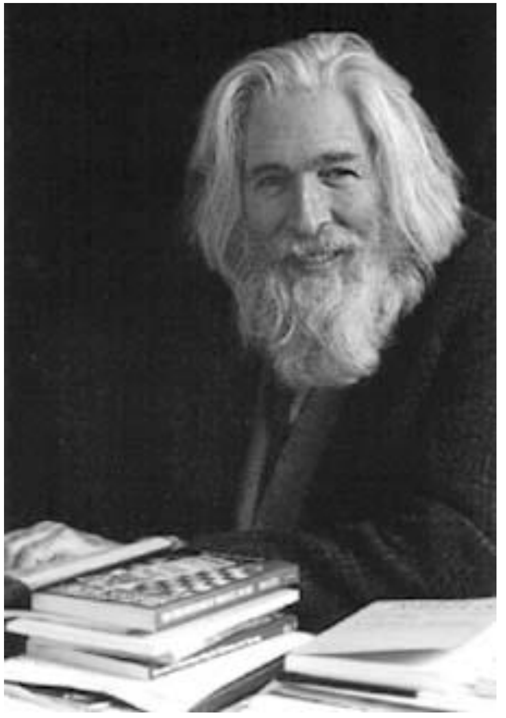


Photo by Paul Halmos

- This Man Needs No Title -- though a couple of his books do. ☺
- But he has one anyway, in fact a distinguished ranks title: emeritus Oscar R. Ewing Professor of Philosophy, Indiana University Bloomington.
- I apologize for recruiting him away from CUNY.

Preamble

"What a lot of books!" she screamed. "And have you really read them all, Monsieur Bonnard?"

"Alas! I have," I replied, "and that is just the reason that I do not know anything; for there is not a single one of those books which does not contradict some other book; so that by the time one has read them all one does not know what to think about anything. That is just my condition, Madame."

- Anatole France, *The Crime of Sylvestre Bonnard,* 1917 (translated by Lafacdio Hearn) . I owe this quote to Jon Doyle.

Outline of Talk

- Review of Belnap-Dunn 4-valued logic
- Applications, esp. WWW
- Uncertainty vs. probability
- Jøsang's "Subjective Logic"
- Two kinds of uncertainty – ignorance vs. conflicting information
- Extension of Jøsang into which the 4-valued logic may be embedded
- Analytic tableaux for the related 3- and 4-valued entailments ("Coupled Trees")
- "Probabilistic" entailments and related 3- and 4-valued entailments

De Morgan Lattice (Monteiro 1960)

- $(D, \leq, \wedge, \vee, \sim)$
- $(D, \leq, \wedge, \vee, \sim)$ is a distributive lattice

  $\leq$ is a partial order on D

  $a \wedge b = \text{glb}\,\{a, b\}$

  $a \vee b = \text{lub}\,\{a, b\}$

  $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

- $a \leq b$ implies $\sim b \leq \sim a$
- $\sim\sim a = a$

Fact (De Morgan Laws): $\sim(a \wedge b) = \sim a \vee \sim b$

$\sim(a \vee b) = \sim a \wedge \sim b$

Dunn's 1966/69 Representation

- Let U be a non-empty set (elements were "topics" in 1966, "situations" in 1969/76, "information states" now).
- A proposition surrogate is a pair of subsets (X^+, X^-) of U

$$(X^+, X^-) \leq (Y^+, Y^-) \text{ iff } X^+ \subseteq Y^+ \;\&\; Y^- \subseteq X^-$$

$$\sim (X^+, X^-) = (X^-, X^+)$$

$$(X^+, X^-) \wedge (Y^+, Y^-) = (X^+ \cap Y^+, X^- \cup Y^-)$$

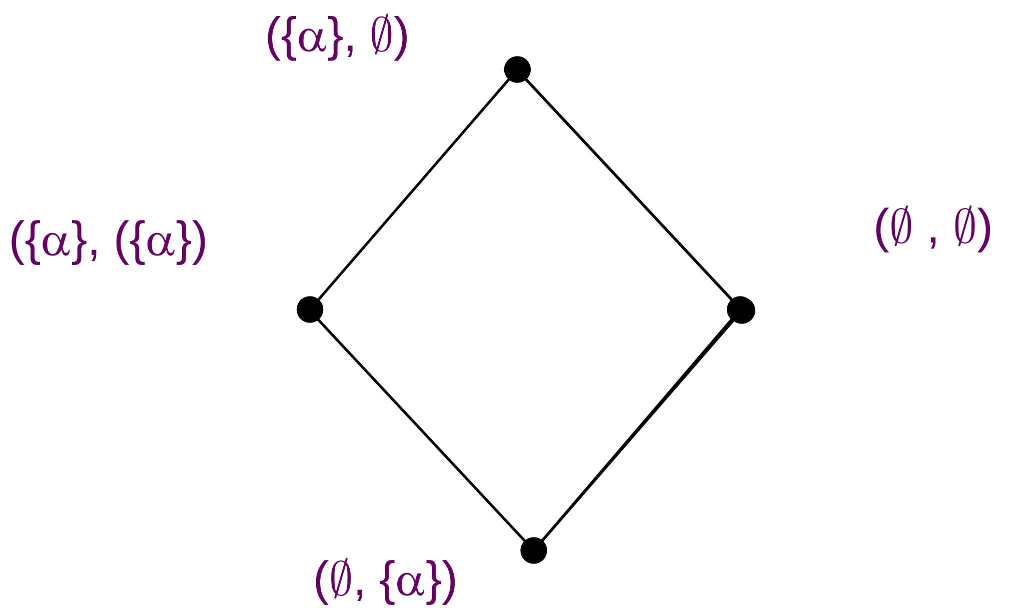
$$(X^+, X^-) \vee (Y^+, Y^-) = (X^+ \cup Y^+, X^- \cap Y^-)$$

Note that X^+, X^- need not exhaust U, and they can overlap.

Theorem: Every De Morgan lattice is isomorphic to a De Morgan lattice of proposition surrogates.

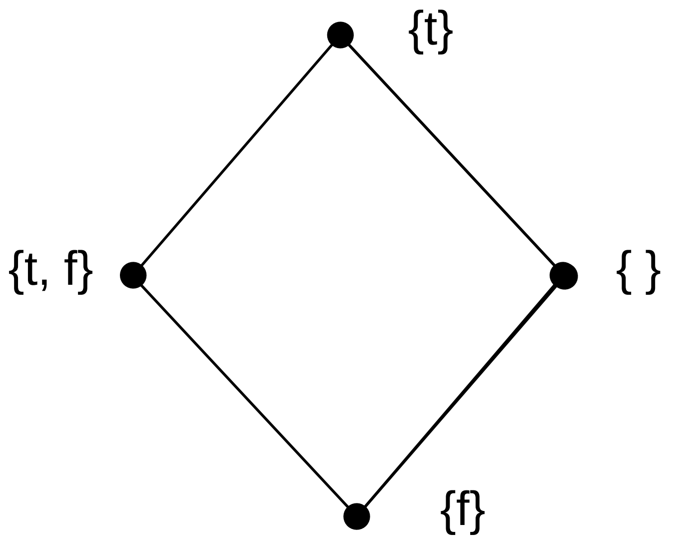
DM4

Dunn (1966/69)

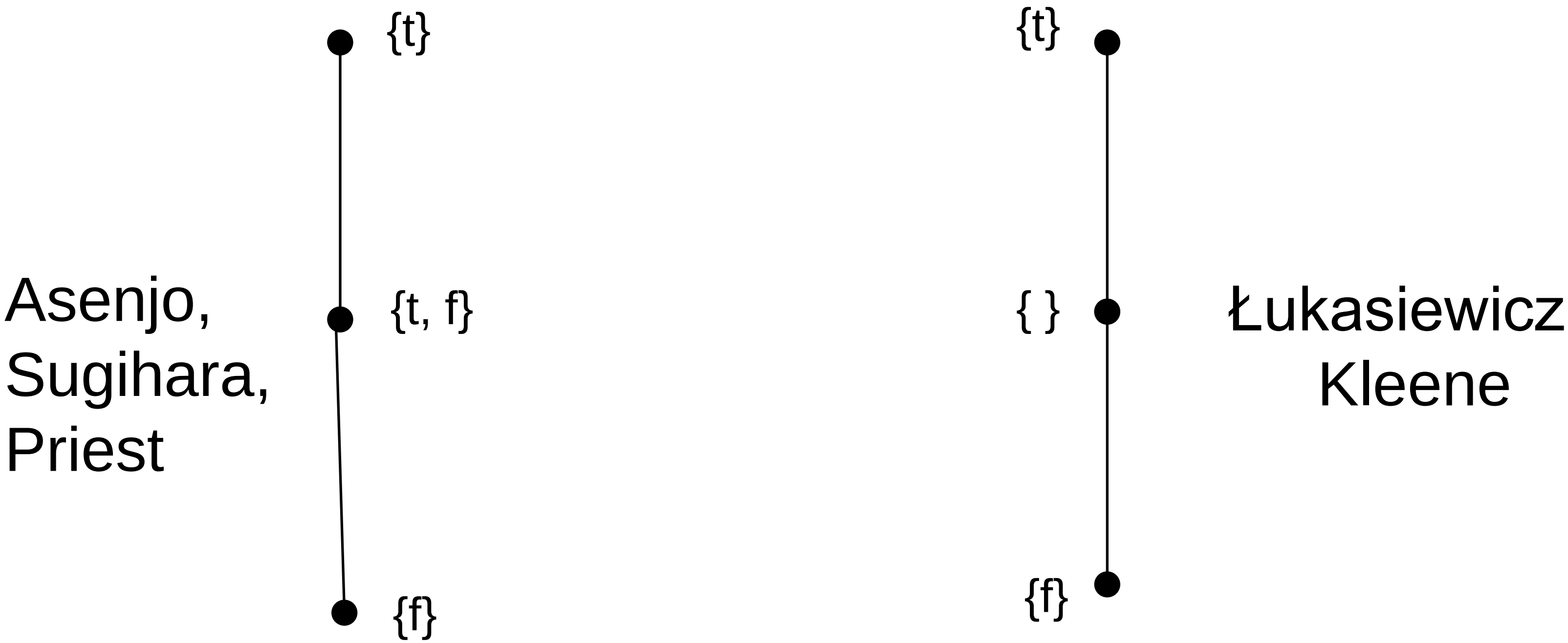


DM4

Dunn (1969, 1976)

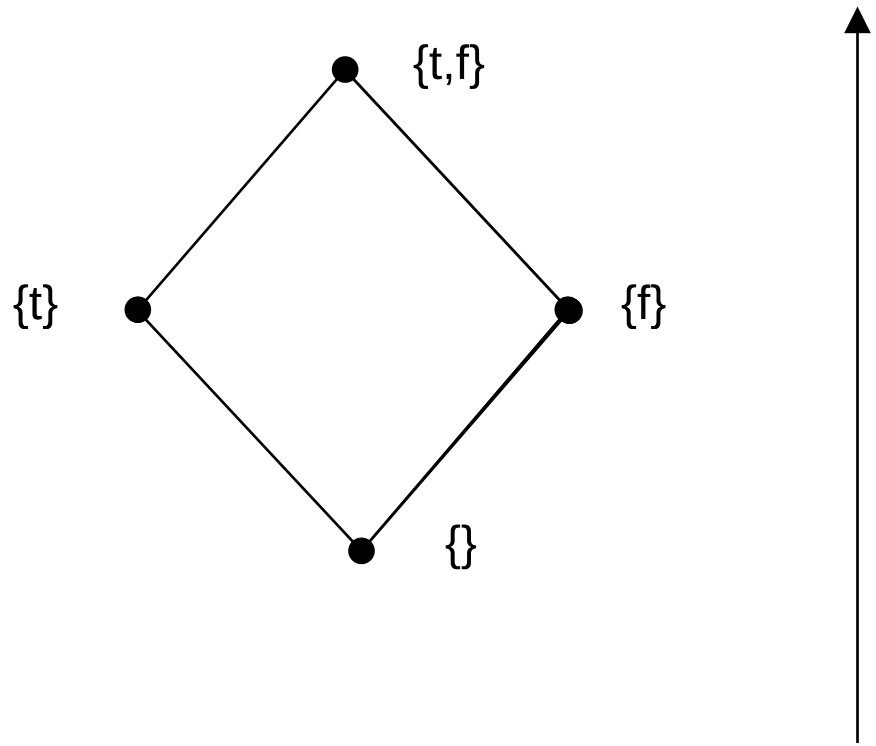


Two Versions of DM3



A4

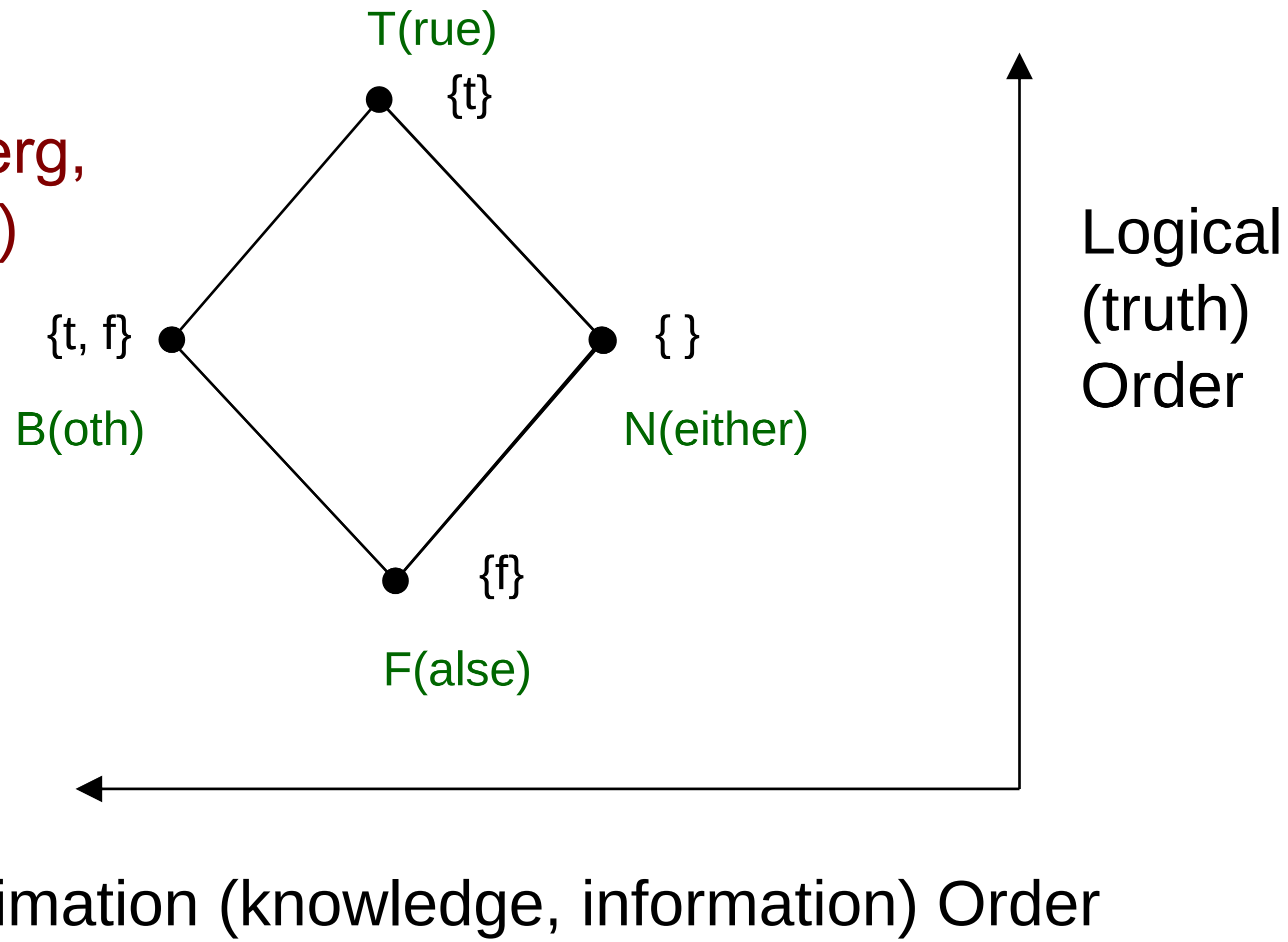
Scott (1970, 72, 73), Belnap (1977, 78)



Approximation (knowledge, information) Order

DM4
as Belnap's L4 (1977, 78)

"Bilatttice" (Ginsberg, Fitting, Avron, etc.)



Applications

- Belnap envisaged DM4 as applying to databases with multiple custodians, where one custodian enters P and another enters not-P. I had envisaged a person being told one thing by one source and the opposite by another
- Belnap’s application might seem essentially the same as mine, but it is in fact much deeper, and was made, presciently, before unstructured databases were in wide use.

- What was once the exception is now the rule – think of the Web, Wikipedia, etc.
- W3C (World Wide Web Consortium) and Tim Berners Lee evisage a time when Web sites will be written in RDF and automated logic will be applied to the Web by intelligent agents. They worry about the disasters that would happen if they use classical logic since then a contradiction implies everything.

Next a seeming digression

Probability vs. Risk: The current economic crisis– somehow I can't get it off my mind. ☺

Frank H. Knight, *Risk, Uncertainty, and Profit*, 1921

Knight distinguished "uncertainty" from "risk". According to Knight, "risk" refers to a situation in which the probability of an outcome can be determined, say using a history of frequency. Therefore the outcome can be rationally insured against. (Risk also typically has a utility built into it alongside probability, but here the focus is on the probability aspect.)

"Uncertainty," by contrast, refers to an event whose probability cannot be known.

In common parlance, uncertainty is often conflated with chance, but So there are at least two kinds of "uncertainty". Consider the toss of a coin.

1) We can quantify the uncertainty of the coin coming up Heads – it would naturally be thought to be 1 in 2, or 0.5. This is the familiar determination of probability.

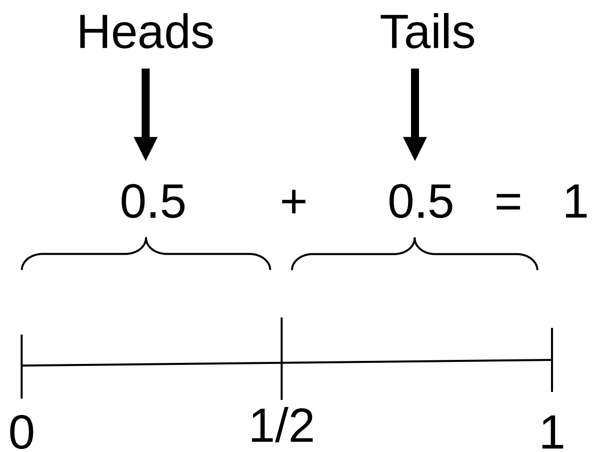
2) But we can worry about the information on which this is based. Maybe the coin is not fair? If we have tossed the coin many times and the two sides come up approximately the same number of times, we have more information that leads to 0.5, or if we have seen how it is made, can measure its balance, know the person tossing the coin to be scrupulously honest, etc. In short, the more information that we have that the coin is fair, the more confidence we have in the probability 0.5.

But suppose we have only incomplete information relevant to the fairness of the coin. E.g., the coin tosser shows us the results only 1/2 of the time, and the other 1/2 keeps the results covered with his hand. If the decision to not uncover them was entirely random this would not change the probabilities. But what if the coin tosser is Ray Smullyan and the decision to not uncover them is far from random?

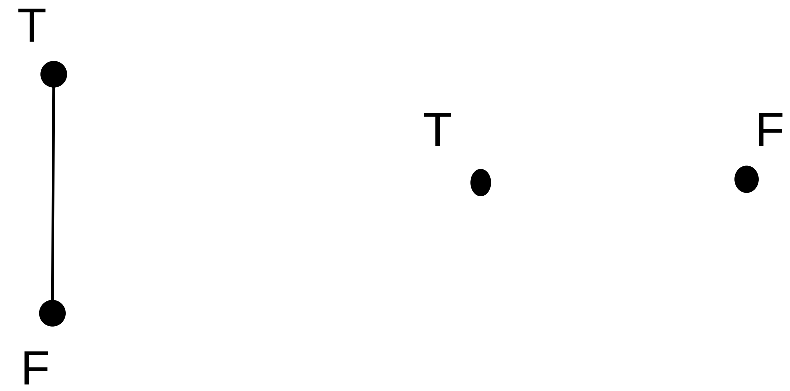
And to look ahead a bit, what if Ray after some of the throws, uncovers the coin and it is Heads, then covers it and again uncovers it, and the coin is Tails? Here we have conflicting information. This is yet a third kind of "uncertainty." But following Jøsang we shall not think of classical probabilities as "uncertain," because they are definite. That still leaves us with two kinds of "uncertainty": (1) ignorance which comes from too little information and (2) conflict which comes from too much information.

We shall first examine (1).

Probabilities



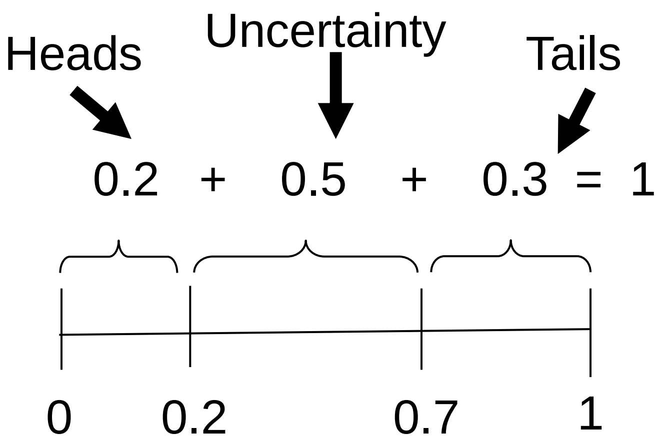
Truth Values



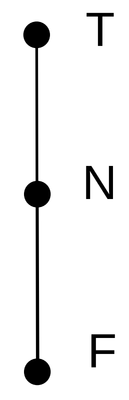
Logical Order

Info Order

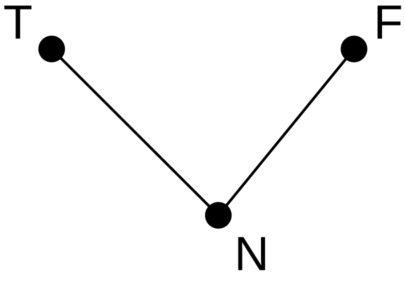
“Probabilities” with Uncertainty



“Truth Values” with Uncertainty



Logical Order



Info Order

"Subjective Logic"

- A. Jøsang, "Artificial Reasoning with Subjective Logic," *Proceedings of the Second Australian Workshop on Commonsense Reasoning*, Perth 1997.
- A. Jøsang, " An Algebra for Assessing Trust in Certification Chains," *Proceedings of the NDSS'99 Network and Distributed Systems Security Symposium*, The Internet Society, San Diego 1999.

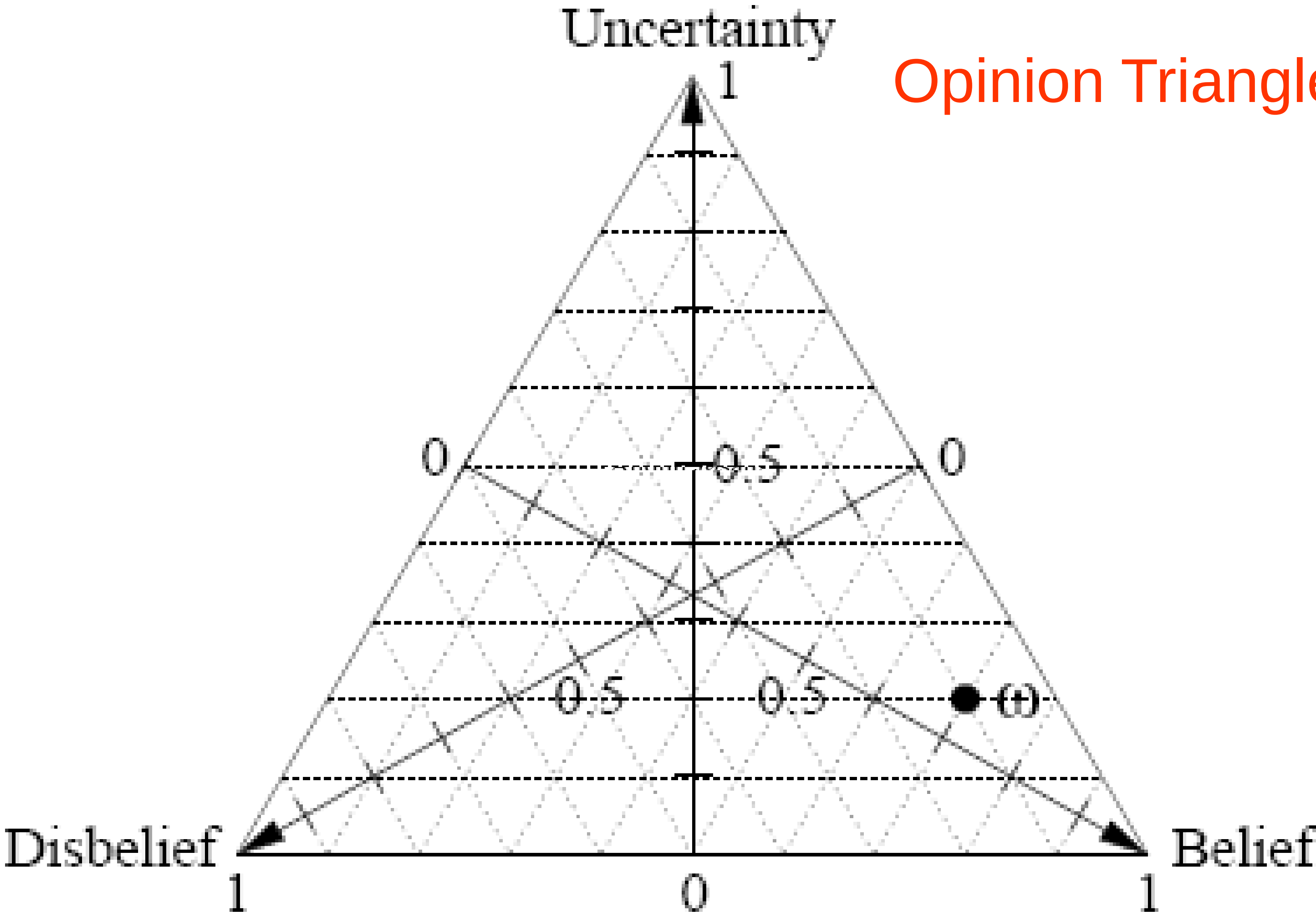
From Wikipedia:

Subjective logic is a type of probabilistic logic that explicitly takes uncertainty and belief ownership into account. In general, subjective logic is suitable for modeling and analysing situations involving uncertainty and incomplete knowledge[1][2]. For example, it can be used for modeling trust networks and for analysing Bayesian networks.

……………….

- A fundamental aspect of the human condition is that nobody can ever determine with absolute certainty whether a proposition about the world is true or false. In addition, whenever the truth of a proposition is expressed, it is always done by an individual, and it can never be considered to represent a general and objective belief.

Opinion Triangle



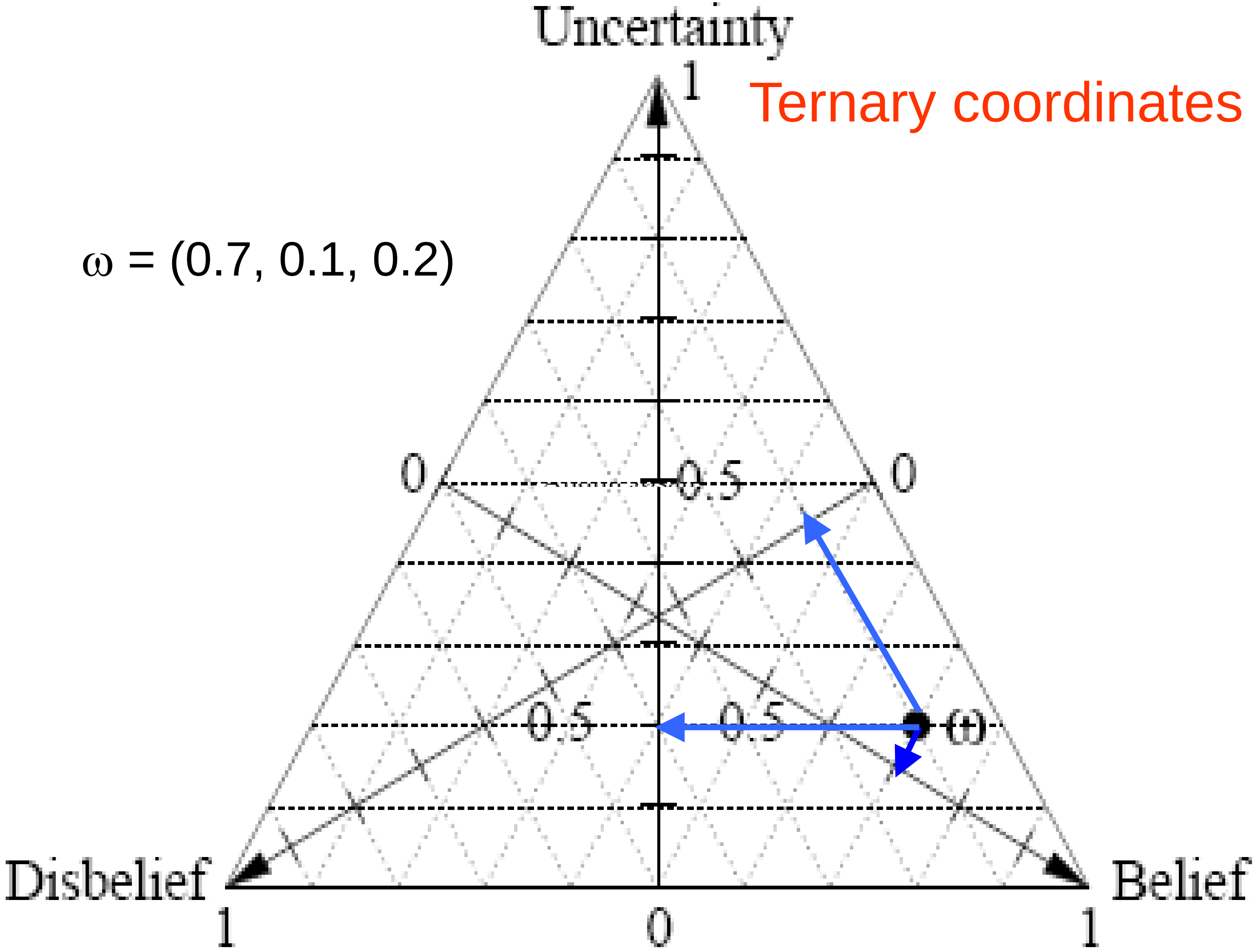
- If we assign p a triple (b, d, 0) this is to say that we are inclined, with no uncertainty, to believe p to degree b and to believe not-p to degree d. This is the “binary case” where we can simply have an “Opinion Line Segment” with degrees b, d between 0 and 1 which always sum to 1.
- Imagine a fair coin toss (and that you are somehow certain that it is fair). Let p = the coin comes up heads. The corresponding triple is (½, ½, 0).

A model for quantifying uncertainty

- It may be apocryphal, but I have heard that in the early days of the U.S. Weather Service, the chance of “rain” was determined by taking a vote of the weathermen. Suppose 10 weathermen are polled, 8 say yes, 2 say no. Then the probability of rain was reported as 0.8.
- But suppose that 7 say yes, 1 says no, and 2 hesitate to offer an opinion. We could interpret this as the degree of belief is 0.7, the degree of disbelief is 0.1, and the degree of uncertainty is 0.2.
- Both of these are particular kinds of normalized weightings according to number of sources.

Ternary coordinates

ω = (0.7, 0.1, 0.2)



In Subjective Logic, conjunction is defined:

$(bp, dp, up) \wedge (bq, dq, uq) =$

$(bp \bullet bq,\ dp + dq - dp \bullet dq,\ bp \bullet uq + up \bullet bq + up \bullet uq).$

Can also define negation and disjunction:

$\sim (bp, dp, up) = (dp, bp, up)$

$(bp, dp, up) \vee (bq, dq, uq) =$

$(bp + bq - bp \bullet bq,\ dp \bullet dq,\ bp \bullet uq + up \bullet bq + up \bullet uq)$

Deconstructing the equation: $(bp, dp, up) \wedge (bq, dq, uq)$

$$=$$

$$bp \bullet bq,\ dp + dq - dp \bullet dq,\ bp \bullet uq + up \bullet bq + up \bullet uq$$

(1) (2) (3)

- Note that the first term (1) behaves like the probability of a conjunction, assuming that p and q are independent.
- Note that the second term (2) behaves like the probability of a disjunction, but the subtracted term $dp \bullet dq$ again assumes that p and q are independent.
- (1) and (2) could be rewritten in a familiar fashion using conditional probability, but that would prolong the talk.

- The third term (3) ($bp \bullet uq + up \bullet bq + up \bullet uq$) has some independent motivation -- the three ways that one can be uncertain about a conjunction, in analogy with:

$$T \wedge N = N \wedge T = N \wedge N = N.$$

- But it can also be obtained purely algebraically as a corrective factor just by assuming that all three terms must sum to 1 and solving the equation:

$$bp \bullet bq + dp + dq - dp \bullet dq + x = 1, \text{ i.e.,}$$

$$x = 1 - bp \bullet bq - dp - dq + dp \bullet dq$$

To make thing look more like high school algebra we write b_1 instead of bp, b_2 instead of bq, etc.

We show that $X = (u_1 b_2 + b_1 u_2 + + u_1 u_2)$ is the solution.

1) $(b_1 + d_1 + u_1)(b_2 + d_2 + u_2) = 1$. So by distributing:

2) $b_1 b_2 + b_1 d_2 + b_1 u_2 + d_1 b_2 + d_1 d_2 + b_1 u_2 + u_1 b_2 + u_1 d_2 + u_1 u_2 = 1$

3) $b_1 b_2 + d_1 + d_2 - d_1 d_2 + \overbrace{u_1 b_2 + b_1 u_2 + + u_1 u_2}^{X} =$

$d_1(b_2 + d_2 + u_2)$ $\quad$ $(b_1 + d_1 + u_1)\, d_2$ $\quad$ Again by distributing:

4) $= b_1 b_2 + (d_1 b_2 + d_1 d_2 + d_1 u_2) + (b_1 d_2 \cancel{+ d_1} d_2 + u_1 d_2) \cancel{- d_1} d_2 + (u_1 b_2 + b_1 u_2 + u_1 u_2) =$

5) = left hand side of 2) = 1.

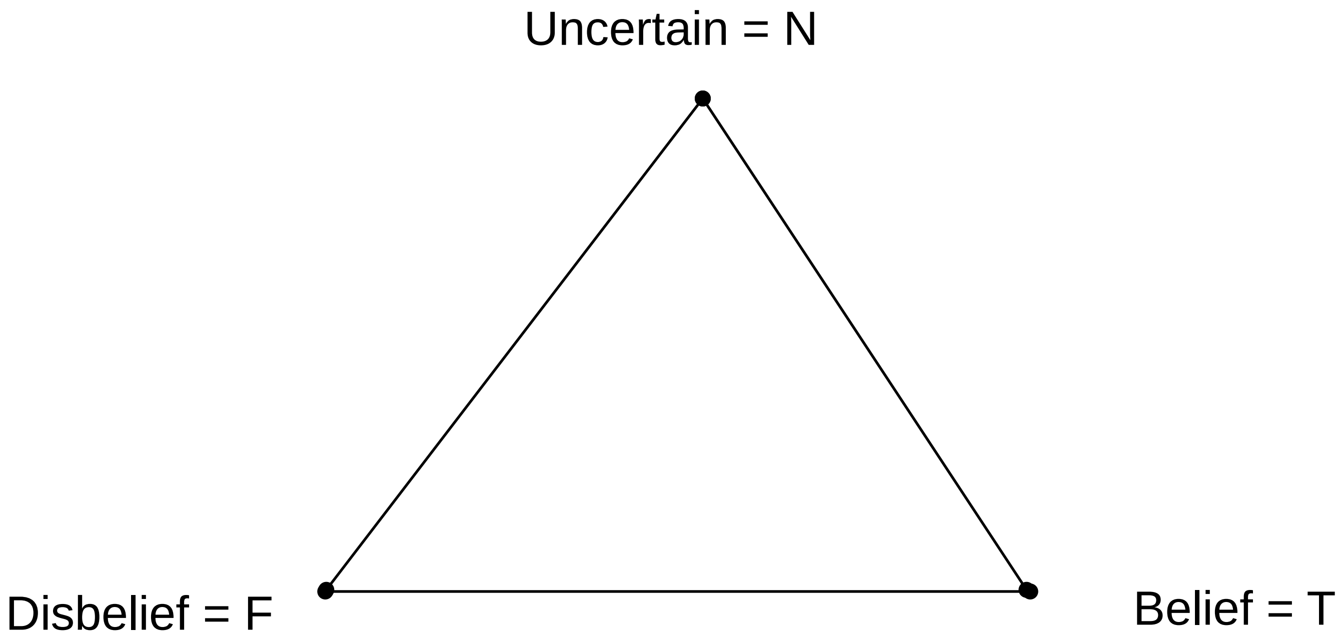
FACT

Set:

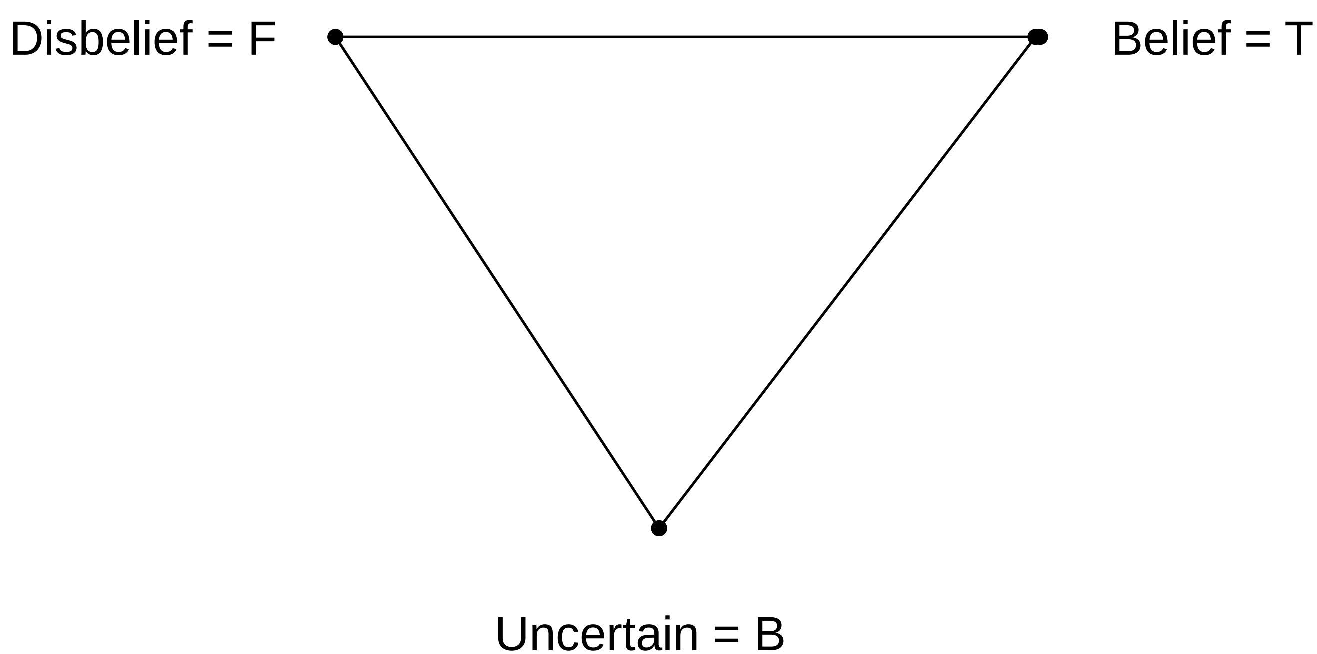
- T = (1,0,0)
- N = (0,0,1)
- F = (0,1,0)

Then $\wedge$**,** $\vee$**,** **~** behave as on DM4 (alternatively as on the strong Kleene algebra, or the 3-valued Łukaziewics algebra, or the 3-valued Sugihara algebra).

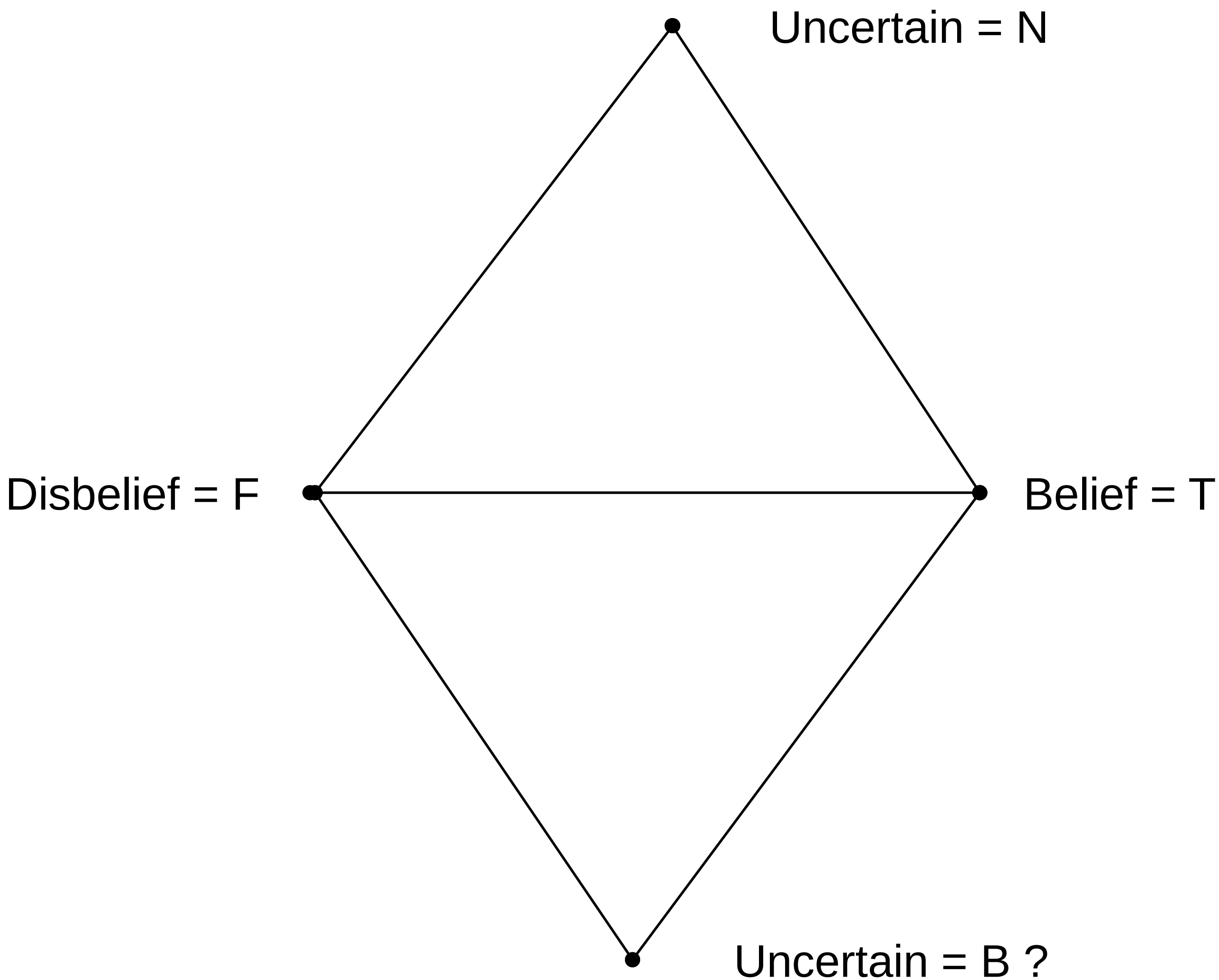
Analogy with right half of DM4



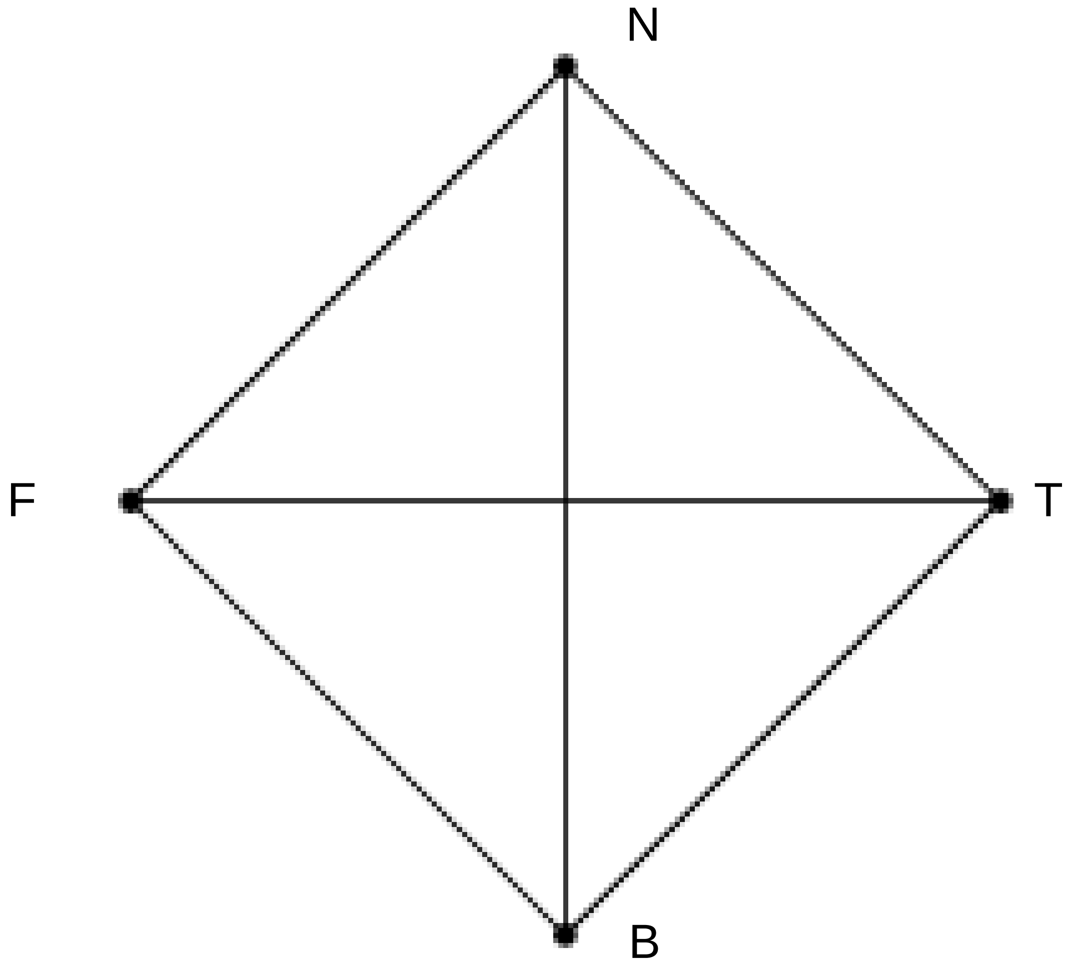
Analogy with left half of DM4



Gluing the analogies together



Let's add a line and visualize it as a tetrahedron!



“Uncertainty” is ambiguous between two different causes: too little information, or too much information

- Everyone has personally experienced the difference between uncertainty caused by lack of information and uncertainty caused by conflicting information.

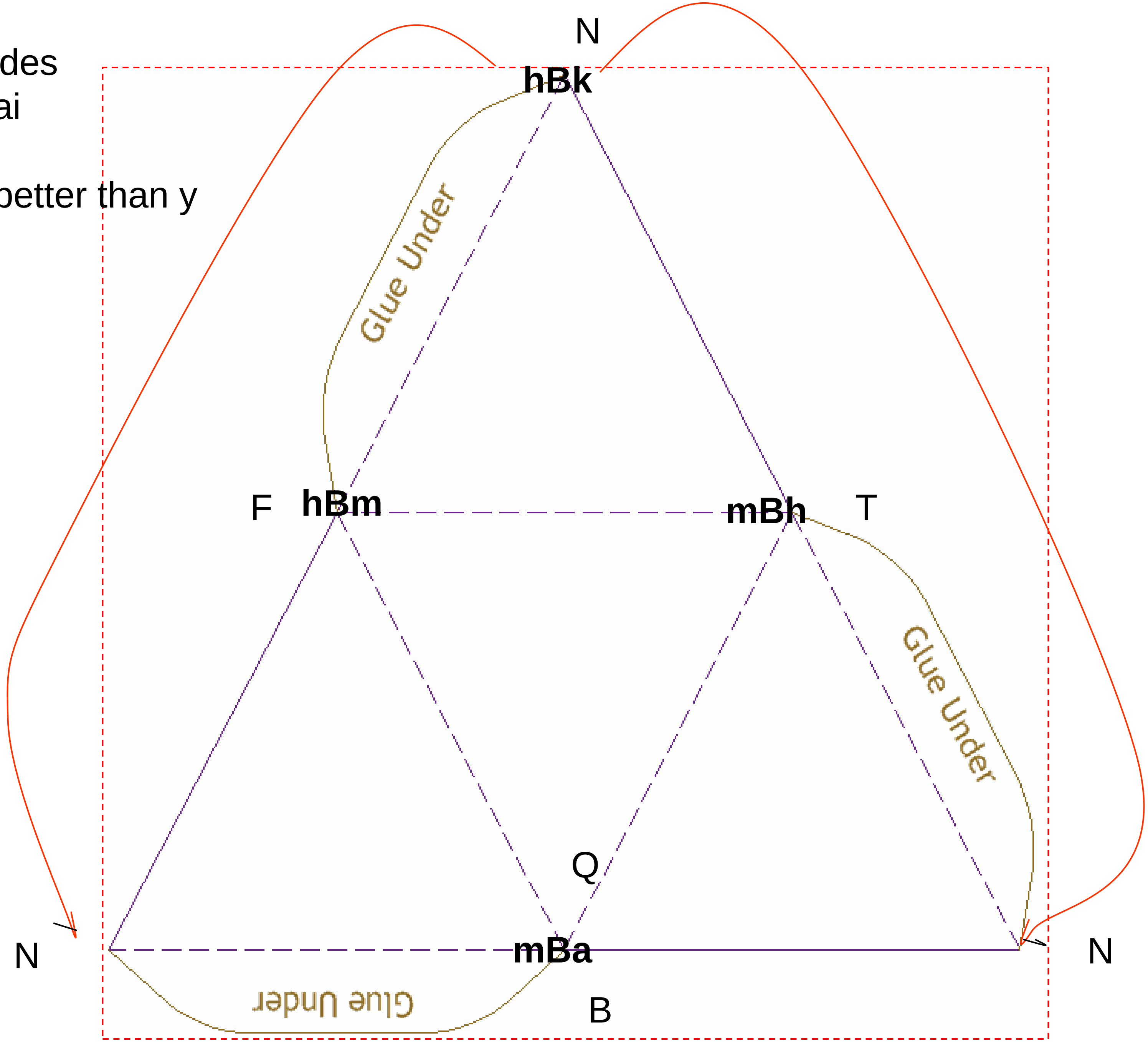
- Imagine the typical German consumer “Otto” trying to decide between a Hyundai and a Kia, and then again between a Mercedes and an Audi.

- Let hBk = Hyundai is better than Kia.
  mBa = Mercedes is better than Audi.
  mBh = Mercedes is better than Hyundai
  hBm = Hyundai is better than Mercedes

- Otto has little if any information regarding the truth of hBk (N), and conflicting information regarding the truth of mBa (B). He of course believes mBh (T) and disbelieves hBm (F).

- The tetrahedon allows us to diagram and quantify the different epistemic circumstances:

a: Audi
m: Mercedes
h: Hyundai
k: Kia
xBy: x is better than y



How to make an “Opinion Tetrahedron”

- Fold so as to make a regular tetrahedron. Drop altitudes from each vertex to the center of the opposite side and by convention assign each the length 1.0 (measuring from 0 at the base to 1 at the vertex). They intersect at 0.25.

A model for quantifying "uncertainty" of two different kinds?

- Let us return to the weathermen where there was some degree of uncertainty. Remember that 7 said yes to rain, 1 said no, and 2 hesitated to offer an opinion. But suppose this time the weathermen could not just abstain – they could also vote both yes and no. Perhaps 1 of them had been in meetings and had no chance to study the prospects of rain, while 1 had worked very hard and had produced persuasive evidence on both sides of the question. We could interpret this as the degree of belief is 0.7, the degree of disbelief is 0.1, the degree of "uncertainty" in the sense of ignorance is 0.1, and the degree of "uncertainty" in the sense of conflict is 0.1.

Meaning of a point on the Opinion Tetrahedron

- A point (b, d, u, c) on the surface of the Opinion Tetrahedron is to be understood as follows: b = degree of belief, d = degree of disbelief, u = degree of uncertainty (ignorance), c = degree of contradiction (inconsistency). $0 \leq b, d, u, c \leq 1$.
- Note: We use “c” for “conflicted” or “contradictory,” instead of “b” for Both, since “b” has already been taken up for belief. I could rewrite this using “T” for “b”, “F” for “d”, “N” for “u”, and “B” for “c”, but I won’t bother to confuse you since time is pressing.

Question

- Can we define conjunction, disjunction, negation on the unit regular tetrahedron so that when we set
- T = (1,0,0,0)
- N = (0,0,1,0)
- B = (0,0,0,1)
- F = (0,1,0,0),

then $\wedge$, $\vee$, ~ behave as on DM4?

- Answer: Generalize the definitions for the Opinion Triangle in the "obvious way."

  $(b_1, d_1, u_1, c_1) \wedge (b_2, d_2, u_2, c_2) =$

  $(b_1 b_2,$

  $d_1 + d_2 - d_1 d_2 + c_1 u_2 + u_1 c_2,$

  $u_1 b_2 + b_1 u_2 + u_1 u_2,$

  $b_1 c_2 + c_1 b_2 + c_1 c_2)$

- Why is this the "obvious way"?

  Because it can be proven that

$b_1 b_2 + d_1 + d_2 - d_1 d_2 + c_1 u_2 + u_1 c_2 + u_1 b_2 + b_1 u_2 + u_1 u_2 + b_1 c_2 + c_1 b_2 + c_1 c_2 = 1$.

$c_1 u_2 + u_1 c_2$ is added to the second component because

$B \wedge N = N \wedge B = F$. The third and fourth components are similarly motivated, respectively, by

$T \wedge N = N \wedge T = N \wedge N = N$,

$T \wedge B = B \wedge T = B \wedge B = B$.

Axioms/Rules for first-degree relevant entailments (R_{fde}):

(A1) $A \wedge B \vdash A$ (A2) $A \wedge B \vdash B$ $\wedge$-elimination

(A3) $A \vdash A \vee B$ (A4) $B \vdash A \vee B$ $\vee$-introduction

(A5) $A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)$ Distribution

(A6) $A \dashv\vdash {\sim\sim} A$ Double Negation

(A7) ${\sim}(A \wedge B) \dashv\vdash {\sim} A \vee {\sim} B$ Negated Conjunction

(A8) ${\sim}(A \vee B) \dashv\vdash {\sim} A \wedge {\sim} B$ Negated Disjunction

(R1) From $A \vdash B$ and $B \vdash C$, infer $A \vdash C$ Transitivity

(R2) From $A \vdash B$ and $A \vdash C$, infer $A \vdash B \wedge C$ $\wedge$ -introduction

(R3) From $A \vdash C$ and $B \vdash C$, infer $A \vee B \vdash C$ $\vee$ -elimination

Supplementary Axioms

(S1) $A \wedge \sim A \;\; \vdash \; B$ (Absurdity) .
(S2) $A \;\; \vdash B \; \vee \sim B$ (Triviality):
(S3) $A \; \wedge \sim A \; \vdash B \; \vee \sim B$ (Safety)

- Adding Absurdity by itself gives a first-degree entailment system we call $KŁ_{fde}$. It is the first-degree entailment fragment of the 3-valued logics of Kleene (1952) and Łukaziewicz (1920).

- Adding Triviality by itself gives the system LP_{fde} of Priest.

- Adding both Absurdity and Triviality at the same time gives a entailment system TV_{fde} for the usual two-valued classical logic.

- Adding Safety gives the first-degree fragment of the semi-relevance logic R-Mingle (RM_{fde}).

- Note that these axiomatizations lack the rule

From $A \vdash B$ infer $\sim B \vdash \sim A$ Contraposition.

This is good because of $KŁ_{fde}$ and LP_{fde}. The first has Absurdity and the last Triviality, but neither has both. One would give the other if they had contraposition.

But it can be proved to be admissible for R_{fde}, TV_{fde}, RM_{fde}.

Consequence relations on DM4:

(i) $A \;{}_{\mathrm{t}}{\models^{BN}} B$ iff, $\forall v$, if $t \in v(A)$ then $t \in v(B)$
(ii) $A \;{}_{\mathrm{f}}{\models^{BN}} B$ iff, $\forall v$, if $f \in v(B)$ then $f \in v(A)$
(iii) $A \;{}_{tf}{\models^{BN}} B$ iff, both $A \;{}_{\mathrm{t}}{\models^{BN}} B$
& $A \;{}_{\mathrm{f}}{\models^{BN}} B$

The superscript ^{BN} is to remind us that this is a valuation into DM4, which contains both of the values B and N. The subscripts indicate whether the consequence relation is truth-preserving (from right-to-left), falsity preserving (from left-to-right), or both.

Of course we can have valuations which stay on just one side of DM4, or the other, and these are valuations respectively into 3B and 3N. Consequence relations are determined as above, and we label them with just a superscript ^{B} or ^{N} accordingly.

(i') $A \;_{\mathrm{t}}\!\models^B B$ (i'') $A \;_{t}\!\models^N B$

(ii') $A \;_{\mathrm{f}}\!\models^B B$ (ii'') $A \;_{f}\!\models^N B$

(iii') $A \;_{tf}\!\models^B B$ (iii'') $A \;_{tf}\!\models^N B$

Dunn (1976) shows that the 3 consequence relations on DM4 are all (extensionally) identical (i.e., they determine the same pairs of sentences) and Dunn (2000) adds that the following pairs are also identical:

$$_{\mathrm{t}}\models^{B} \quad \text{with} \quad _{f}\models^{N},$$

$$_{\mathrm{f}}\models^{B} \quad \text{with} \quad _{t}\models^{N},$$

$$_{tf}\models^{B} \quad \text{with} \quad _{tf}\models^{N}.$$

Completeness Theorems for these Axiomatizations [Dunn (2000)]

(i) The system R_{fde} is characterized by any of the (extensionally) identical entailment relations $_t|\!=^{BN}$, $_f|\!=^{BN}$, $_{tf}|\!=^{BN}$ B.

(ii) the system $KŁ_{fde}$ by $_f|\!=^{B}$ (equals $_t|\!=^{N}$)

(ii) the system LP_{fde} by $_t|\!=^{B}$ (equals $_f|\!=^{N}$)

(iv) the system RM_{fde} by $_{tf}|\!=^{B}$ (equals $_{tf}|\!=^{N}$)

We show how to characterize these logics using something like the Analytic Tableaux of Smullyan (1968)

Rules for Smullyan's Analytic Tableaux:

$\sim\sim A$	$A \wedge B$	$\sim(A \wedge B)$	$A \vee B$	$\sim(A \vee B)$
A	A	$\sim A$ | $\sim B$	A | B	$\sim A$
	B			$\sim B$

Sample Tableau Proof

Show: $\sim A \wedge \sim B \mid\!- \sim(A \vee B)$

- $\sim A \wedge \sim B$ Hyp
- $\sim\sim(A \vee B)$ Hyp for Reductio ad Absurdum
- $A \vee B$ 2, $\sim\sim$-elim
- A B 3, $\vee$-elim
- $\sim A$ $\sim B$ 1, $\wedge$-elim

X X Both branches “close”

Coupled Trees
R. Jeffrey (1967)

Build a tableaux down from the premiss and up from the conclusion, e.g.:

Show: $\sim A \vee \sim B$ |- $\sim(A \wedge B)$

- $\sim A \vee \sim B$ Hyp
- $\sim A$ $\sim B$ 1, $\wedge$-elim

↓ ↓

2’. $\sim A$ $\sim B$

1’. $\sim(A \wedge B)$

Another Example of Coupled Trees

Show: $\sim A \wedge (A \vee B) \mid\!- B$

1. $\sim A \wedge (A \vee B)$

2. $\sim A$ 1, $\wedge$-elim

3. $A \vee B$

4. A B

?

1'. B

Coupled Trees

Show: $\sim A \wedge (A \vee B) \vdash B$

1. $\sim A \wedge (A \vee B)$

2. $\sim A$ 1, $\wedge$-elim

3. $A \vee B$

4. A B

X

Just in time!

1'. B

Add these rules to get TV_{fde}

"Forfeit"

$$\begin{array}{c} \bullet \\ A \\ \sim A \\ \text{X} \end{array}$$

"Punt"

$$\begin{array}{ccc} & \bullet & \\ A & & \sim A \end{array}$$

$A \wedge \sim A \ \vdash \ B$

$B \ \vdash \ A \vee \sim A$

Playing with the rules Forfeit and Punt we can get these many-valued logics:

- TV_{fde}: Has both Forfeit and Punt (Jeffrey 1967)
- R_{fde}: Add neither Forfeit nor Punt (Dunn 1976)
- $KŁ_{fde}$: Add just Forfeit
- LP_{fde}: Add just Punt
- RM_{fde}: Only a Forfeited branch can be extended by Punt.

Probability valuations

- A (probability) valuation is a function *P* on sentences (whose connectives are just $\sim$, $\vee$, $\wedge$) to real numbers such that
- (A1) $0 \leq P(A) \leq 1$;
- (A2) $P(A) = 1$ if A is a tautology;
- (A3) $P(A \vee B) = P(A) + P(B)$ when *A* and *B* are logically incompatible, i.e., $\sim(A \wedge B)$ is a tautology.

Assigning Probabilities to Compound Sentences

From these axioms the standard rules for assigning probabilities to compound sentences can be derived:

(P1) $P(\sim A) = 1 - P(A);$

(P2) $P(A \wedge B) = P(A) \times P(B/A)$, where $P(B/A)$ is the probability of *B* given *A*, defined in usual way.

(P3) $P(A \vee B) = P(A) + P(B) - P(A \wedge B)$

Note that when A and B are independent, (P2) and (P3) can be simplified:

(P2') $P(A \wedge B) = P(A) \times P(B)$

(P3') $P(A \vee B) = P(A) + P(B) - (P(A) \times P(B))$

Classical Probabilistic Entailment

How then do we define consequence using probabilities? The obvious probabilistic way to define that A entails B is as follows:

$$\forall P,\ P(A) \leq P(B).$$

The probability of B is always at least as high as the probability of A.

(P1) allows us to show that if $A \models B$, then $A \models_{prob} B$.

Proof. If if $A \models B$ then $\sim A \vee B$ is a tautology. Hence $P(\sim A \vee B) = 1$. (A2)

$P(\sim A) + P(B) - P(A \wedge B) = 1$ (P2)

$1 - P(A) + P(B) - P(A \wedge B) = 1$ (P1)

$1 + P(B) - P(A \wedge B) = 1 + P(A)$ Add $P(A)$

$P(B) - P(A \wedge B) = P(A)$ Subtract 1

$P(A) \leq P(B)$

“Probabilistic” Consequence relations using Jøsang’s Opinion Triangle with U = N

(i) $A \,_{\mathrm{b}}|\!=^{N} \; B$ iff $\forall v$, $v_{\mathrm{b}}\,(A) \leq v_{\mathrm{b}}(B)$

Belief in B is always at least as strong as belief in A.

(ii) $A \,_{\mathrm{d}}|\!=^{N} \; B$ iff $\forall v$, $v_{\mathrm{d}}\,(B) \leq v_{\mathrm{d}}\,(A)$

Disbelief in A is always at least as strong as disbelief in B.

(iii) $A \,_{bd}|\!=^{N} B$ iff both $A \,_{\mathrm{b}}|\!=^{BN} \; B$

$A \,_{\mathrm{d}}|\!=^{BN} \; B$

- We can similarly define “probabilistic” consequence relations using the Dual Opinion Triangle (corresponding to U = B) and the Opinion Tetrahedron (which corresponds to having both B and N).

We assume that these probabilistic entailment relations include their logical counterparts, i. e.,

$$_{\mathrm{t}}|\!=^{N} \subseteq {}_{\mathrm{b}}|\!=^{N}, \qquad {}_{\mathrm{f}}|\!=^{N} \subseteq {}_{\mathrm{d}}|\!=^{N}$$

$$_{\mathrm{t}}|\!=^{B} \subseteq {}_{\mathrm{b}}|\!=^{B}, \qquad {}_{\mathrm{f}}|\!=^{B} \subseteq {}_{\mathrm{d}}|\!=^{B}$$

$$_{\mathrm{t}}|\!=^{BN} \subseteq {}_{\mathrm{b}}|\!=^{BN}, \qquad {}_{\mathrm{f}}|\!=^{BN} \subseteq {}_{\mathrm{d}}|\!=^{BN}$$

Logical Conclusion

Then we get completeness theorems for Hence we get both soundness and completeness for each of the systems TV_{fde}, $KŁ_{fde}$, LP_{fde}, RM_{fde} and R_{fde} with respect to the appropriate probabilistic. entailment.

And hence we get a coupled tree version of Smullyan's analytic tableaux for each of these "probabilistic" systems.

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- Thank you, and especially **thank you Raymond!**